International Journal of Modern Physics C, Vol. 9, No. 7 (1998) 899–902

A SIMPLE CELLULAR AUTOMATON THAT SOLVES THE DENSITY AND ORDERING PROBLEMS

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Received 10 July 1998
Revised 27 July 1998

We show that there exists a simple solution to the density problem in cellular automata, under fixed boundary conditions, in contrast to previously used periodic ones.

Keywords: Cellular Automata; Density Problem; Emergent Computation.

Cellular automata (CA) are discrete, dynamical systems that perform computations in a distributed fashion on a spatially extended grid. The dynamical behavior of a CA may give rise to emergent computation, referring to the appearance of global information processing capabilities that are not explicitly represented in the system's elementary components nor in their local interconnections.[1] As such, CAs offer an austere yet versatile model for studying natural phenomena, as well as a powerful paradigm for attaining fine-grained, massively parallel computation.

An example of such emergent computation is to use a CA to determine the global density of bits in an initial state configuration. This problem, known as density classification, has been studied quite intensively over the past few years. In this short communication we describe two previous versions of the problem along

with their CA solutions, and then go on to show that there exists yet a third version — which admits a simple solution.

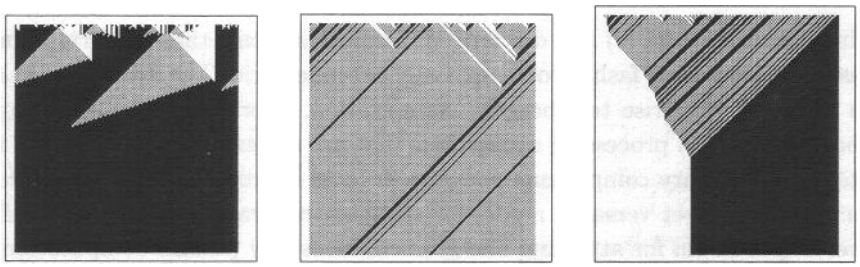
Version I. In the original statement of the problem,[2] a one-dimensional, two-state CA is presented with an arbitrary initial configuration of states (the input), and should converge in time to a state of all 1s if the initial configuration contains a density of 1s > 0.5, and to all 0s if this density < 0.5; for an initial density of 0.5, the CA's behavior is undefined (Fig. 1(I)). The final configuration is considered as the output of the computation. Spatially periodic boundary conditions are used, resulting in a circular grid. Land and Belew[3] proved that for a one-dimensional grid of fixed size N, and for a fixed radius $r \geq 1$, there exists no two-state CA rule which correctly solves this problem version, i.e., correctly classifies all possible initial configurations. Recently, researchers have focused on the use of artificial evolution techniques, demonstrating that high-performance CAs can be evolved to solve this version of the problem.[4–7] These CAs do not perform perfect classification, i.e., they misclassify some of the initial configurations (the CA solution demonstrated in Fig. 1(I) does not in fact classify correctly all initial configurations).

Version II. Capcarrère, Sipper, and Tomassini[8] showed that a perfect one-dimensional, two-state, radius $r = 1$ CA density classifier does exist, upon defining a different output specification. The CA rule in question is defined as follows:

$$s_i(t+1) = \begin{cases} s_{i-1}(t) & \text{if } s_i(t) = 0 \\ s_{i+1}(t) & \text{if } s_i(t) = 1 \end{cases},$$

where $s_i(t)$ is the state of cell i at time t (this rule is numbered 184 in Wolfram's CA numbering scheme[9]). Again, periodic boundary conditions are assumed.

Upon presentation of an arbitrary initial configuration, the N-cell grid relaxes to a limit-cycle, within $\lceil N/2 \rceil$ time steps, that provides a classification of the initial configuration's density of 1s: if this density > 0.5 (respectively, < 0.5), then the final configuration consists of one or more blocks of at least two consecutive 1s (0s),



(I) (II) (III)

Fig. 1. CA solutions to three versions of the density classification problem. Grid size is $N = 149$. White squares represent cells in state 0, black squares represent cells in state 1. The pattern of state configurations is shown through time (which increases down the page). The initial density in all three examples is > 0.5.

interspersed by an alternation of 0s and 1s; for an initial density of exactly 0.5, the final configuration consists of an alternation of 0s and 1s. The computation's output is given by the state of the consecutive block (or blocks) of same-state cells (Fig. 1(II)). Capcarrère, Sipper, and Tomassini[8] proved that this rule performs perfect density classification (including the density=0.5 case). They also established that the seemingly more complex output specification is actually as simple as the previous one. (Interestingly, it has recently been shown that one can obtain the output specified in Version I by applying rule 184 for $N/2$ time steps, followed by application of rule 232 for $N/2$ time steps.[10])

Version III. We now describe yet another modification of the original problem (version I), with (1) a different output specification, as well as (2) *fixed boundary conditions*, rather than the periodic ones previously assumed. These two modifications give rise to a simple density classifier. Consider a one-dimensional, two-state, $r = 1$, rule-184 CA with fixed boundary cells: the left cell is fixed at state 0, and the right cell is fixed at state 1. The finite grid of size N (boundary cells excluded) will converge in at most $N - 1$ time steps to a configuration $0^\alpha 1^\beta$, where α, β denote the number of 0s and 1s at time step 0, respectively; $\alpha, \beta \in \{0, \ldots, N\}$, $\alpha + \beta = N$. In the cases where N is odd, the density classification of the input is attained by considering the middle cell's final state: 0 signifies a majority of 0s in the input, 1 signifies a majority of 1s; for N even, we consider the two middle cells: 00 signifies a majority of 0s in the input, 11 signifies a majority of 1s, 01 signifies equality, and 10 is impossible (Fig. 1(III)). To show that the above property holds for the fixed-boundaries, rule-184 CA we proceed in two stages.

(1) The density of a configuration does not change from time step t to $t + 1$. To see this we express the 8 next-state bits of rule 184 (i.e., the rule table) as follows: (a) $00x \mapsto 0$, (b) $x11 \mapsto 1$, (c) $x10 \mapsto 0$, (d) $10x \mapsto 1$, where $x \in \{0, 1\}$. For each cell, exactly one of the four possibilities applies at time step t: (a) and (b) do not change the cell's state, thereby causing no change in density, while (c) applies iff (d) applies to the adjacent cell, resulting again in density unchanged. (This argument also holds for the cells adjacent to the border cells.)

(2) As long as there is a cell pair with states 10 then the configuration is "unstable" — the next time step will result in the interchange of both cells' states, i.e., the 1 "moves" to the right and the 0 moves to the left. From the rule one can directly observe that a 1 cannot travel to the left and a 0 cannot travel to the right. Essentially, there is a constant flow of 1s to the right, as long as a 10 pair exists. The boundary conditions act to "block" this flow of 0s and 1s (as opposed to periodic conditions where these continue to cycle throughout the grid). Thus, given density conservation, one can see that the only stable configuration, toward which the CA converges, is $0^\alpha 1^\beta$. It is straightforward to see that the worst-case convergence time is $N - 1$.

We note that, serendipitously, this CA also solves another, seemingly more difficult problem, known as ordering.[5] This is a sorting problem in which the CA must place all 0s on the left and all 1s on the right, i.e., converge toward the $0^{\alpha}1^{\beta}$ configuration. Sipper[5] showed that no one-dimensional, $r = 1$ CA with periodic boundary conditions can solve this problem.

In summary, we have shown that by changing the output specification *and* the boundary conditions, with respect to the original problem statement, a simple density classifier can be attained, as well as a sorter.

Density is a global property of a configuration (the 1s can be distributed throughout the grid), whereas a small-radius CA relies solely on local interactions. This holds true for all three versions of the problem, yet its solutions can be either impossible (version I) or easy (versions II and III); thus, density in itself is not an intrinsically hard problem to compute. This raises the general issue of identifying intrinsically hard problems for such local systems, and distinguishing them from those that can be transformed into easy problems.

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